

Stationary Algorithmic Probability

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Abstract—Kolmogorov complexity and algorithmic probability are defined only up to an additive resp. multiplicative constant, since their actual values depend on the choice of the universal reference computer. In this paper, we analyze a natural approach to eliminate this machine-dependence.

Our method is to assign algorithmic probabilities to the different computers themselves, based on the idea that “unnatural” computers should be hard to emulate. Therefore, we study the Markov process of universal computers randomly emulating each other. The corresponding stationary distribution, if it existed, would give a natural and machine-independent probability measure on the computers, and also on the binary strings.

Unfortunately, we show that no stationary distribution exists on the set of all computers; thus, this method cannot eliminate machine-dependence. Moreover, we show that the reason for failure has a clear and interesting physical interpretation, suggesting that every other conceivable attempt to get rid of those additive constants must fail in principle, too.

However, we show that restricting to some subclass of computers might help to get rid of some amount of machine-dependence in some situations, and the resulting stationary computer and string probabilities have beautiful properties.

Index Terms—Algorithmic Probability, Kolmogorov Complexity, Markov Chain, Emulation, Emulation Complexity

I. INTRODUCTION AND MAIN RESULTS

SINCE algorithmic probability has first been studied in the 1960s by Solomonoff, Levin, Chaitin and others (cf. [1], [2], [3]), it has revealed a variety of interesting properties, including applications in computer science, inductive inference and statistical mechanics (cf. [4], [5], [6]). The algorithmic probability of a binary string s is defined as the probability that a universal prefix computer U outputs s on random input, i.e.

$$P_U(s) := \sum_{x \in \{0,1\}^*: U(x)=s} 2^{-|x|}, \quad (1)$$

where $|x|$ denotes the length of a binary string $x \in \{0,1\}^*$. It follows from the Kraft inequality that

$$\sum_{s \in \{0,1\}^*} P_U(s) =: \Omega_U < 1,$$

where Ω_U is Chaitin’s famous halting probability. So algorithmic probability is a subnormalized probability distribution or semimeasure on the binary strings. It is closely related to prefix Kolmogorov complexity $K_U(s)$ which is defined [4] as the length of the shortest computer program that outputs s :

$$K_U(s) := \min\{|x| \mid U(x) = s\}.$$

The relation between the two can be written as

$$K_U(s) = -\log P_U(s) + \mathcal{O}(1), \quad (2)$$

where the $\mathcal{O}(1)$ -term denotes equality up to an additive constant. Both Kolmogorov complexity and algorithmic probability depend on the choice of the universal reference computer U . However, they do not depend on U “too much”: If U and V are both universal prefix computers, then it follows from the fact that one can emulate the other that

$$K_U(s) = K_V(s) + \mathcal{O}(1),$$

i.e. the complexities K_U and K_V differ from each other only up to an additive constant. Then Equation (2) shows that the corresponding algorithmic probabilities differ only up to a multiplicative constant.

This kind of “weak” machine independence is good enough for many applications: if the strings are long enough, then a fixed additive constant does not matter too much. However, there are many occasions where it would be desirable to get rid of those additive constants, and to eliminate the arbitrariness which comes from the choice of the universal reference computer. Examples are Artificial Intelligence [6] and physics [7], where one often deals with finite and short binary strings.

We start with a simple example, to show that the machine-dependence of algorithmic probability can be drastic, and also to illustrate the main idea of our approach. Suppose that U_{nice} is a “natural” universal prefix computer, say, one which is given by a Turing machine model that we might judge as “simple”. Now choose an arbitrary strings s consisting of a million random bits; say, s is attained by a million tosses of a fair coin. With high probability, there is no short program for U_{nice} which computes s (otherwise toss the coin again and use a different string s). We thus expect that

$$P_{U_{\text{nice}}}(s) \approx 2^{-1.000.000}.$$

Now we define another prefix computer U_{bad} as

$$U_{\text{bad}}(x) := \begin{cases} s & \text{if } x = 0, \\ \text{undefined} & \text{if } x = \lambda \text{ or } x = 0y, \\ U_{\text{nice}}(y) & \text{if } x = 1y. \end{cases}$$

The computer U_{bad} is universal, since it emulates the universal computer U_{nice} if we just prepend a “1” to the input. Since $U_{\text{bad}}(0) = s$, we have

$$P_{U_{\text{bad}}}(s) > \frac{1}{2}.$$

Hence the algorithmic probability $P_U(s)$ depends drastically on the choice of the universal computer U . Clearly, the computer U_{bad} seems quite unnatural, but in algorithmic information theory, all the universal computers are created equal — there is no obvious way to distinguish between them and to say which one of them is a “better choice” than the other.

So what is “the” algorithmic probability of the single string s ? It seems clear that $2^{-1.000.000}$ is a better answer than $\frac{1}{2}$, but the question is how we can make mathematical sense of this statement. How can we give a sound formal meaning to the statement that U_{nice} is more “natural” than U_{bad} ? A possible answer is that in the process of randomly constructing a computer from scratch, one is very unlikely to end up with U_{bad} , while there is some larger probability to encounter U_{nice} .

This suggests that we might hope to find some natural probability distribution μ on the universal computers, in such a way that $\mu(U_{\text{bad}}) \ll \mu(U_{\text{nice}})$. Then we could define the “machine-independent” algorithmic probability $P(s)$ of some string s as the weighted average of all algorithmic probabilities $P_U(s)$,

$$P(s) := \sum_{U \text{ universal}} \mu(U) P_U(s). \quad (3)$$

Guided by Equation (2), we could then define “machine-independent Kolmogorov complexity” via $K(s) := -\log P(s)$.

But how can we find such a probability distribution μ on the computers? The key idea here is to compare the capabilities of the two computers to emulate each other. Namely, by comparing U_{nice} and U_{bad} , one observes that

- it is very “easy” for the computer U_{bad} to emulate the computer U_{nice} : just prepend a “1” to the input. On the other hand,
- it is very “difficult” for the computer U_{nice} to emulate U_{bad} : to do the simulation, we have to supply U_{nice} with the long string s as additional data.

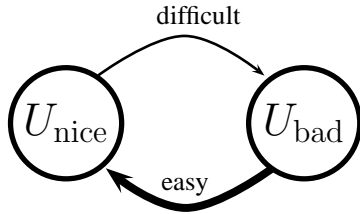


Fig. 1. Computers that emulate each other.

The idea is that this observation holds true more generally: “Unnatural” computers are harder to emulate. There are two obvious approaches to construct some computer probability μ from this observation — interestingly, both turn out to be equivalent:

- The situation in Figure 1 looks like the graph of some Markov process. If one starts with either one of the two computers depicted there and interprets the line widths as transition probabilities, then in the long run of more and more moves, one tends to have larger probability to end up at U_{nice} than at U_{bad} . So let’s apply this idea more generally and define a Markov process of all the universal computers, randomly emulating each other. If the process has a stationary distribution (e.g. if it is positive recurrent), this is a good candidate for computer probability.
- Similarly as in Equation (1), there should be a simple way to define probabilities $P_U(V)$ for computers U and

V , that is, the probability that U emulates V on random input. Then, whatever the desired computer probability μ looks like, to make any sense, it should satisfy

$$\mu(U) = \sum_{V \text{ universal}} \mu(V) P_V(U).$$

But if we enumerate all universal computers as $\{U_1, U_2, U_3, \dots\}$, this equation can be written as

$$\begin{pmatrix} \mu(U_1) \\ \mu(U_2) \\ \mu(U_3) \\ \dots \end{pmatrix} = \begin{pmatrix} P_{U_1}(U_1) & P_{U_2}(U_1) & \dots \\ P_{U_1}(U_2) & P_{U_2}(U_2) & \dots \\ P_{U_1}(U_3) & P_{U_2}(U_3) & \dots \\ \dots & \dots & \dots \end{pmatrix} \cdot \begin{pmatrix} \mu(U_1) \\ \mu(U_2) \\ \mu(U_3) \\ \dots \end{pmatrix}$$

Thus, we should look for the unknown stationary probability eigenvector $\underline{\mu}$ of the “emulation matrix” $(P_{U_i}(U_j))_{i,j}$.

Clearly, both ideas are equivalent if the probabilities $P_U(V)$ are the transition probabilities of the aforementioned Markov process.

Now we give a synopsis of the paper and explain our main results:

- Section II contains some notational preliminaries, and defines the *output frequency* of a string as the frequency that this string is output by a computer. For prefix computers, this notion equals algorithmic probability (Example 2.2).
- In Section III, we define the emulation Markov process that we have motivated above, and analyze if it has a stationary distribution or not. Here is the construction for the most important case (the case of the full set of computers) in a nutshell: we say that a computer C *emulates* computer D via the string x , and write $C \xrightarrow{x} D$ and $D = \left(C \xrightarrow{x}\right)$ if $C(xy) = D(y)$ for all strings y . A computer is *universal* if it emulates every other computer. Given a universal computer, at least one of the two computers $C \xrightarrow{0}$ and $C \xrightarrow{1}$ must be universal, too.

Thus, we can consider the universal computers as the vertices of a graph, with directed edges going from U to V if $U \xrightarrow{0} V$ or $U \xrightarrow{1} V$. Every vertex (universal computer) has either one or two outgoing edges (corresponding to the two bits). The random walk on this connected graph defines a Markov process: we start at some computer, follow the outgoing edges, and if there are two edges, we follow each of them with probability $\frac{1}{2}$. This is schematically depicted in Figure 2.

If this process had a stationary distribution, this would be a good candidate for a natural algorithmic probability measure on the universal computers. Unfortunately, no stationary distribution exists: this Markov process is transient.

We prove this in Theorem 3.13. The idea is to construct a sequence of universal computers M_1, M_2, M_3, \dots such that M_i emulates M_{i+1} with high probability — in fact, with probability turning to 1 fast as i gets large. The corresponding part of the emulation Markov process is depicted in Figure 3. The outgoing edges in the upwards

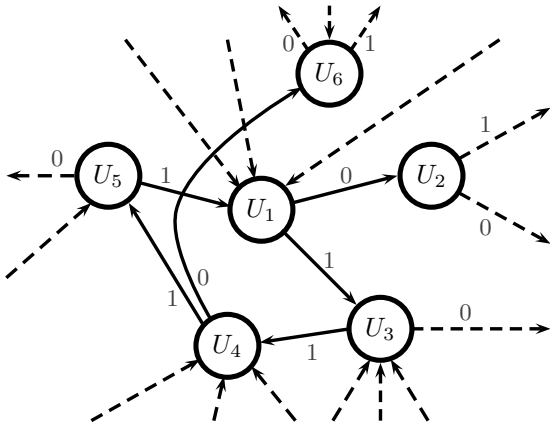


Fig. 2. Schematic diagram of the emulation Markov process. Note that the zeroes and ones represent input bits, *not* transition probabilities, for example $U_1(1y) = U_3(y)$ for every string y . In our notation, we have for example $U_1 \xrightarrow{110} U_6$.

direction lead back to a fixed universal reference computer, which ensures that every computer M_i is universal.

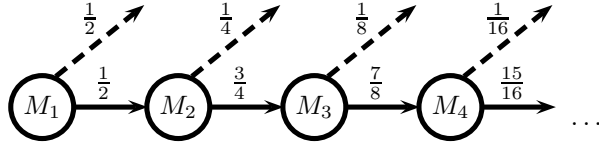


Fig. 3. This construction in Theorem 3.13 proves that the emulation Markov chain is transient, and no stationary distribution exists (the numbers are transition probabilities). It is comparable to a “computer virus” in the sense that each M_i emulates “many” slightly modified copies of itself.

As our Markov process has only transition probabilities $\frac{1}{2}$ and 1, the edges going from M_i to M_{i+1} in fact consist of several transitions (edges). As those transition probabilities are constructed to tend to 1 very fast, the probability to stay on this M_i -path forever (and not return to any other computer) is positive, which forces the process to be transient.

Yet, it is still possible to construct analogous Markov processes for restricted sets of computers Φ . Some of those sets yield processes which have stationary distributions; a non-trivial example is given in Example 3.14.

- For those computer sets Φ with positive recurrent emulation process, the corresponding computer probability has nice properties that we study in Section IV. The computer probability induces in a natural way a probability distribution on the strings $s \in \{0, 1\}^*$ (Definition 4.1) as the probability that the random walk described above encounters some output which equals s . This probability is computer-independent and can be written in several equivalent ways (Theorem 4.2).
- A symmetry property of computer probability yields another simple and interesting proof why for the set of all computers — and for many other natural computer sets — the corresponding Markov process cannot be positive recurrent (Theorem 4.7). In short, if σ is a computable

permutation, then a computer C and the output permuted computer $\sigma \circ C$ must have the same probability as long as both are in the computer set Φ (Theorem 4.6). If there are infinitely many of them, they all must have probability zero which contradicts positive recurrence.

- For the same reason, there cannot be one particular “natural” choice of a computer set Φ with positive recurrent Markov process, because $\sigma \circ \Phi$ is always another good (positive recurrent) candidate, too (Theorem 4.8).
- This has a nice physical interpretation which we explain in Section V: algorithmic probability and Kolmogorov complexity always contain at least the ambiguity which is given by permuting the output strings. This permutation can be interpreted as “renaming” the objects that the strings are describing.

We argue that this kind of ambiguity will be present in any attempt to eliminate machine-dependence from algorithmic probability or complexity, even if it is different from the approach in this paper. This conclusion can be seen as the main result of this work.

Finally, we show in the appendix that the string probability that we have constructed equals, under certain conditions, the weighted average of output frequency — this is a particularly unexpected and beautiful result (Theorem A.6) which needs some technical steps to be proved. The main tool is the study of input transformations, i.e., to permute the strings before the computation. The appendix is the technically most difficult part of this paper and can be skipped on first reading.

II. PRELIMINARIES AND OUTPUT FREQUENCY

We start by fixing some notation. In this paper, we only consider *finite, binary strings*, which we denote by

$$\{0, 1\}^* := \bigcup_{n=0}^{\infty} \{0, 1\}^n = \{\lambda, 0, 1, 00, 01, \dots\}.$$

The symbol λ denotes the *empty string*, and we write the *length* of a string $s \in \{0, 1\}^*$ as $|s|$, while the cardinality of a set S is denoted $\#S$. To avoid confusion with the composition of mappings, we denote the *concatenation* of strings with the symbol \otimes , e.g.

$$101 \otimes 001 = 101001.$$

In particular, we have $|\lambda| = 0$ and $|x \otimes y| = |x| + |y|$. A *computer* C is a partial-recursive function $C : \{0, 1\}^* \rightarrow \{0, 1\}^*$, and we denote the set of all computers by Ξ . Note that our computers do not necessarily have to have prefix-free domain (unless otherwise stated). If $C \in \Xi$ does not halt on some input $x \in \{0, 1\}^*$, then we write $C(x) = \infty$ as an abbreviation for the fact that $C(x)$ is undefined. Thus, we can also interpret computers C as mappings from $\{0, 1\}^*$ to $\overline{\{0, 1\}^*}$, where

$$\overline{\{0, 1\}^*} := \{0, 1\}^* \cup \{\infty\}.$$

As usual, we denote by $K_C(x)$ the *Kolmogorov complexity* of the string $x \in \{0, 1\}^*$ with respect to the computer $C \in \Xi$

$$K_C(x) := \min \{|s| \mid s \in \{0, 1\}^*, C(s) = x\}$$

or as ∞ if this set is empty.

What would be a first, naive try to define algorithmic probability? Since we do not restrict our approach to prefix computers, we cannot take Equation (1) as a definition. Instead we may try to count how often a string is produced by the computer as output:

Definition 2.1 (Output Frequency):

For every $C \in \Xi$, $n \in \mathbb{N}_0$ and $s \in \{0, 1\}^*$, we set

$$\mu_C^{(n)}(s) := \frac{\#\{x \in \{0, 1\}^n \mid C(x) = s\}}{2^n}.$$

For later use in Section III, we also define for every $C, D \in \Xi$ and $n \in \mathbb{N}_0$

$$\mu_C^{(n)}(D) := \frac{\#\{x \in \{0, 1\}^n \mid C \xrightarrow{x} D\}}{2^n},$$

where the expression $C \xrightarrow{x} D$ is given in Definition 3.1.

Our final definition of algorithmic probability will look very different, but it will surprisingly turn out to be closely related to this output frequency notion.

The existence of the limit $\lim_{n \rightarrow \infty} \mu_C^{(n)}(s)$ depends on the computer C and may be hard to decide, but in the special case of prefix computers, the limit exists and agrees with the classical notion of algorithmic probability as given in Equation (1):

Example 2.2 (Prefix Computers): A computer $C \in \Xi$ is called *prefix* if the following holds:

$$C(x) \neq \infty \implies C(x \otimes y) = \infty \text{ for every } y \neq \lambda.$$

This means that if C halts on some input $x \in \{0, 1\}^*$, it must not halt on any extension $x \otimes y$. Such computers are traditionally studied in algorithmic information theory. To fit our approach, we need to modify the definition slightly. Call a computer $C_p \in \Xi$ *prefix-constant* if the following holds true:

$$C_p(x) \neq \infty \implies C_p(x \otimes y) = C_p(x) \text{ for every } y \in \{0, 1\}^*.$$

It is easy to see that for every prefix computer C , one can find a prefix-constant computer C_p with $C_p(x) = C(x)$ whenever $C(x) \neq \infty$. It is constructed in the following way: Suppose $x \in \{0, 1\}^*$ is given as input into C_p , then it

- computes the set of all prefixes $\{x_i\}_{i=0}^{|x|}$ of x (e.g. for $x = 100$ we have $x_0 = \lambda$, $x_1 = 1$, $x_2 = 10$ and $x_3 = 100$),
- starts $|x| + 1$ simulations of C at the same time, which are supplied with x_0 up to $x_{|x|}$ as input,
- waits until one of the simulations produces an output $s \in \{0, 1\}^*$ (if this never happens, C_p will loop forever),
- finally outputs s .

Fix an arbitrary string $s \in \{0, 1\}^*$. Consider the set

$$T^{(n)}(s) := \{x \in \{0, 1\}^* \mid |x| \leq n, C(x) = s\}.$$

Every string $x \in T^{(n)}(s)$ can be extended (by concatenation) to a string x' of length n . By construction, it follows that $C_p(x') = s$. There are $2^{n-|x|}$ possible extensions x' , thus

$$\mu_{C_p}^{(n)}(s) = \frac{\sum_{x \in T^{(n)}(s)} 2^{n-|x|}}{2^n} = \sum_{x \in \{0, 1\}^* : |x| \leq n, C(x) = s} 2^{-|x|}.$$

It follows that the limit $\mu_{C_p}(s) := \lim_{n \rightarrow \infty} \mu_{C_p}^{(n)}(s)$ exists, and it holds

$$\mu_{C_p}(s) = \sum_{x \in \{0, 1\}^* : C(x) = s} 2^{-|x|},$$

so the output frequency as given in Definition 2.1 converges for $n \rightarrow \infty$ to the classical algorithmic probability as given in Equation (1). Note that $\Omega_C = 1 - \mu_{C_p}(\infty)$. \square

It is easy to construct examples of computers which are *not* prefix, but which have an output frequency which either converges, or at least does not tend to zero as $n \rightarrow \infty$. Thus, the notion of output frequency generalizes the idea of algorithmic probability to a larger class of computers.

III. STATIONARY COMPUTER PROBABILITY

As explained in the introduction, it will be an essential part of this work to analyze in detail how “easily” one computer C emulates another computer D . Our first definition specializes what we mean by “emulation”:

Definition 3.1 (Emulation): A computer $C \in \Xi$ *emulates* the computer $D \in \Xi$ via $x \in \{0, 1\}^*$, denoted

$$C \xrightarrow{x} D \quad \text{resp.} \quad D = \left(C \xrightarrow{x} \right),$$

if $C(x \otimes s) = D(s)$ for every $s \in \{0, 1\}^*$. We write $C \longrightarrow D$ if there is some $x \in \{0, 1\}^*$ such that $C \xrightarrow{x} D$.

It follows easily from the definition that $C \xrightarrow{\lambda} C$ and

$$C \xrightarrow{x} D \text{ and } D \xrightarrow{y} E \implies C \xrightarrow{x \otimes y} E.$$

Now that we have defined emulation, it is easy to extend the notion of Kolmogorov complexity to emulation complexity:

Definition 3.2 (Emulation Complexity): For every $C, D \in \Xi$, the *Emulation Complexity* $K_C(D)$ is defined as

$$K_C(D) := \min \left\{ |s| \mid s \in \{0, 1\}^*, C \xrightarrow{s} D \right\} \quad (4)$$

or as ∞ if the corresponding set is empty.

Note that similar definitions have already appeared in the literature, see for example Def. 4.4 and Def. 4.5 in [8], or the definition of the constant “sim(C)” in [9].

Definition 3.3 (Universal Computer): Let $\Phi \subset \Xi$ be a set of computers. If there exists a computer $U \in \Phi$ such that $U \longrightarrow X$ for every $X \in \Phi$, then Φ is called *connected*, and U is called a Φ -*universal computer*. We use the notation $\Phi^U := \{C \in \Phi \mid C \text{ is } \Phi\text{-universal}\}$, and we write $\overline{\Phi^U} := \{C \in \Xi \mid C \longrightarrow D \ \forall D \in \Phi \text{ and } \exists X \in \Phi : X \longrightarrow C\}$.

Note that $\Phi^U \subset \overline{\Phi^U}$ and $\Phi^U = \emptyset \iff \overline{\Phi^U} = \emptyset$. Examples of connected sets of computers include the set Ξ of all computers and the set of prefix-constant computers, whereas the set of computers which always halt on every input cannot be connected, as is easily seen by diagonalization. For convenience, we give a short proof of the first statement:

Proposition 3.4: The set of all computers Ξ is connected.

Proof. It is well-known that there is a computer U that takes a description $d_M \in \{0, 1\}^*$ of any computer $M \in \Xi$ together with some input $x \in \{0, 1\}^*$ and simulates M on input x , i.e.

$$U(\langle d_M, x \rangle) = M(x) \text{ for every } x \in \{0, 1\}^*,$$

where $\langle \cdot, \cdot \rangle : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ is a bijective and computable encoding of two strings into one. We can construct the encoding in such a way that $\langle d_M, x \rangle = \tilde{d}_M \otimes x$, i.e. the description is encoded into some prefix code that is appended to the left-hand side of x . It follows that $U \xrightarrow{\tilde{d}_M} M$, and since this works for every $M \in \Xi$, U is Ξ -universal. \square

Here is a basic property of Kolmogorov and emulation complexity:

Theorem 3.5 (Invariance of Complexities): Let $\Phi \subset \Xi$ be connected, then for every $U \in \overline{\Phi^U}$ and $V \in \Phi$, it holds that

$$\begin{aligned} K_U(D) &\leq K_U(V) + K_V(D) && \text{for every } D \in \Phi, \\ K_U(s) &\leq K_U(V) + K_V(s) && \text{for every } s \in \{0, 1\}^*. \end{aligned}$$

Proof. Since $U \in \overline{\Phi^U}$, it holds $U \rightarrow V$. Let x be a shortest string such that $U(x \otimes t) = V(t)$ for every $t \in \{0, 1\}^*$, i.e. $|x| = K_U(V)$. If p_D resp. p_s are shortest strings such that $V \xrightarrow{p_D} D$ resp. $V(p_s) = s$, then $|p_D| = K_V(D)$ and $|p_s| = K_V(s)$, and additionally $U \xrightarrow{x \otimes p_D} D$ and $U(x \otimes p_s) = s$. Thus, $K_U(D) \leq |x \otimes p_D| = |x| + |p_D|$ and $K_U(s) \leq |x \otimes p_s| = |x| + |p_s|$. \square

Suppose some computer $C \in \Xi$ emulates another computer E via the string 10, i.e. $C \xrightarrow{10} E$. We can decompose this into two steps: Let $D := C \xrightarrow{1}$, then

$$C \xrightarrow{1} D \quad \text{and} \quad D \xrightarrow{0} E.$$

Similarly, we can decompose every emulation $C \xrightarrow{x} D$ into $|x|$ parts, just by parsing the string x bit by bit, while getting a corresponding “chain” of emulated computers. A clear way to illustrate this situation is in the form of a tree, as shown in Figure 4. We start at the root λ . Since $C \xrightarrow{\lambda} C$, this string corresponds to the computer C itself. Then, we are free to choose 0 or 1, yielding the computer $(C \xrightarrow{0})$ or $(C \xrightarrow{1}) = D$ respectively. Ending up with D , we can choose the next bit (taking a 0 we will end up with $E = (D \xrightarrow{0}) = (C \xrightarrow{10})$) and so on.

In general, some of the emulated computers will themselves be elements of Φ and some not. As in Figure 4, we can mark every path that leads to a computer that is itself an element of Φ by a thick line. (In this case, for example $C, D, E \in \Phi$, but $(C \xrightarrow{11}) \notin \Phi$.) If we want to restrict the process of parsing through the tree to the marked (thick) paths, then we need the following property:

Definition 3.6: A set of computers $\Phi \subset \Xi$ is called *branching*, if for every $C \in \Phi$, the following two conditions are satisfied:

- For every $x, y \in \{0, 1\}^*$, it holds

$$(C \xrightarrow{x \otimes y}) \in \Phi \implies (C \xrightarrow{x}) \in \Phi.$$

- There is some $x \in \{0, 1\}^* \setminus \{\lambda\}$ such that $(C \xrightarrow{x}) \in \Phi$.

If Φ is branching, we can parse through the corresponding marked subtree without encountering any dead end, with the possibility to reach every leaf of the subtree. In particular, these requirements are fulfilled by sets of universal computers:

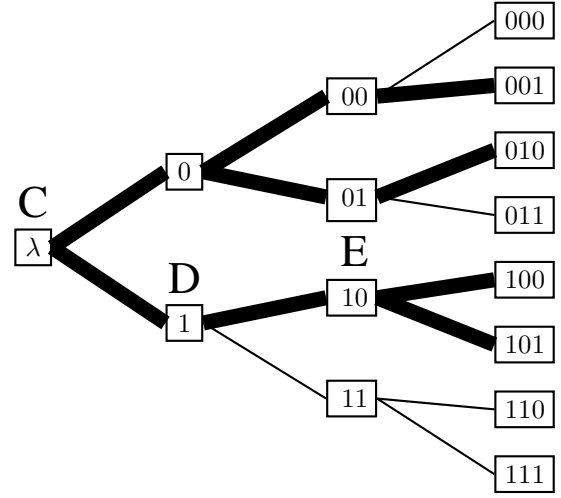


Fig. 4. Emulation as a tree, with a branching subset Φ (bold lines).

Proposition 3.7: Let $\Phi \subset \Xi$ be connected and $\#\overline{\Phi^U} \geq 2$, then $\overline{\Phi^U}$ is branching.

Proof. Let $C \in \overline{\Phi^U}$ and $(C \xrightarrow{x \otimes y}) \in \overline{\Phi^U}$, then $(C \xrightarrow{x \otimes y}) \rightarrow D$ for every $D \in \Phi$, and so $(C \xrightarrow{x}) \rightarrow D$ for every $D \in \Phi$. Moreover, there is some $X \in \Phi$ such that $X \rightarrow C$, so in particular, $X \rightarrow (C \xrightarrow{x})$. Thus, $(C \xrightarrow{x}) \in \overline{\Phi^U}$.

On the other hand, since $\#\overline{\Phi^U} \geq 2$, there are computers $C, D \in \overline{\Phi^U}$ such that $C \neq D$. By definition of $\overline{\Phi^U}$, there is some $X \in \Phi$ such that $X \rightarrow D$. Since C emulates every computer in Φ , we have $C \rightarrow X$, so $C \xrightarrow{z} D$ for some $z \neq \lambda$. \square

As illustrated in the bold subtree in Figure 4, we can define the process of a *random walk* on this subtree if its corresponding computer subset Φ is branching: we start at the root λ , follow the branches, and at every bifurcation, we turn “left or right” (i.e. input an additional 0 or 1) with probability $\frac{1}{2}$. This random walk generates a probability distribution on the subtree:

Definition 3.8 (Path and Computer Probability): If $\Phi \subset \Xi$ is branching and let $C \in \Phi$, we define the Φ -tree of C as the set of all inputs $x \in \{0, 1\}^*$ that make C emulate a computer in Φ and denote it by $C^{-1}(\Phi)$, i.e.

$$C^{-1}(\Phi) := \{x \in \{0, 1\}^* \mid (C \xrightarrow{x}) \in \Phi\}.$$

To every x in the Φ -tree of C , we can associate its *path probability* $\mu_{C^{-1}(\Phi)}(x)$ as the probability of arriving at x on a random walk on this tree. Formally,

$$\begin{aligned} \mu_{C^{-1}(\Phi)}(\lambda) &:= 1, \\ \mu_{C^{-1}(\Phi)}(x \otimes b) &:= \begin{cases} \frac{1}{2} \mu_{C^{-1}(\Phi)}(x) & \text{if } x \otimes \bar{b} \in C^{-1}(\Phi) \\ \mu_{C^{-1}(\Phi)}(x) & \text{otherwise} \end{cases} \end{aligned}$$

for every bit $b \in \{0, 1\}$ with $x \otimes b \in C^{-1}(\Phi)$, where \bar{b} denotes the inverse bit. The associated n -step computer probability of $D \in \Phi$ is defined as the probability of arriving at computer

D on a random walk of n steps on this tree, i.e.

$$\mu_C^{(n)}(D|\Phi) := \sum_{x \in \{0,1\}^n: C \xrightarrow{x} D} \mu_{C^{-1}(\Phi)}(x).$$

It is clear that for $\Phi = \Xi$, we get back the notion of output frequency as given in Definition 2.1: For every $C, D \in \Xi$, it holds

$$\mu_C^{(n)}(D) = \mu_C^{(n)}(D|\Xi).$$

The condition that Φ shall be branching guarantees that $\sum_{x \in \{0,1\}^n \cap C^{-1}(\Phi)} \mu_{C^{-1}(\Phi)}(x) = 1$ for every $n \in \mathbb{N}_0$, i.e. the conservation of probability. For example, the path probability in Figure 4 has values $\mu_{C^{-1}(\Phi)}(0) = \mu_{C^{-1}(\Phi)}(1) = \frac{1}{2}$, $\mu_{C^{-1}(\Phi)}(00) = \mu_{C^{-1}(\Phi)}(01) = \frac{1}{4}$, $\mu_{C^{-1}(\Phi)}(10) = \frac{1}{2}$, $\mu_{C^{-1}(\Phi)}(001) = \mu_{C^{-1}(\Phi)}(010) = \frac{1}{4} = \mu_{C^{-1}(\Phi)}(100) = \mu_{C^{-1}(\Phi)}(101)$.

It is almost obvious that the random walk on the subtree that generates the computer probabilities $\mu_C^{(n)}(D|\Phi)$ is a Markov process, which is the statement of the next lemma. For later reference, we first introduce some notation for the corresponding transition matrix:

Definition 3.9 (Emulation Matrix): Let $\Phi \subset \Xi$ be branching, and enumerate the computers in Φ in arbitrary order: $\Phi = \{C_1, C_2, \dots\}$. Then, we define the (possibly infinite) emulation matrix E_Φ as

$$(E_\Phi)_{i,j} := \mu_{C_i}^{(1)}(C_j|\Phi).$$

Lemma 3.10 (Markovian Emulation Process): If $\Phi \subset \Xi$ is branching, then the computer probabilities $\mu_C^{(n)}(\cdot|\Phi)$ are n -step probabilities of some Markov process (which we also denote Φ) whose transition matrix is given by the emulation matrix E_Φ . Explicitly, with $\delta_i := (0, \dots, 0, \underbrace{1}_i, 0, \dots)$,

$$\left(\mu_{C_i}^{(n)}(C_1|\Phi), \mu_{C_i}^{(n)}(C_2|\Phi), \mu_{C_i}^{(n)}(C_3|\Phi), \dots \right) = \delta_i \cdot (E_\Phi)^n \quad (5)$$

for every $n \in \mathbb{N}_0$, and we have the *Chapman-Kolmogorov equation*

$$\mu_C^{(m+n)}(D|\Phi) = \sum_{X \in \Phi} \mu_C^{(m)}(X|\Phi) \mu_X^{(n)}(D|\Phi) \quad (6)$$

for every $m, n \in \mathbb{N}_0$. Also, $\Phi \subset \Xi$ (resp. E_Φ) is irreducible if and only if $C \longrightarrow D$ for every $C, D \in \Phi$.

Proof. Equation (5) is trivially true for $n = 0$ and is shown in full generality by induction (the proof details are not important for the following argumentation and can be skipped):

$$\begin{aligned} \mu_{C_i}^{(n+1)}(C_j|\Phi) &= \sum_{\substack{x \in \{0,1\}^n, b \in \{0,1\}: \\ C_i \xrightarrow{x \otimes b} C_j}} \mu_{C^{-1}(\Phi)}(x \otimes b) \\ &= \sum_{C_k \in \Phi} \sum_{\substack{x \in \{0,1\}^n: \\ C_i \xrightarrow{x} C_k}} \sum_{\substack{b \in \{0,1\}: \\ C_k \xrightarrow{b} C_j}} \mu_{C^{-1}(\Phi)}(x \otimes b) \\ &= \sum_{C_k \in \Phi} \sum_{\substack{x \in \{0,1\}^n: \\ C_i \xrightarrow{x} C_k}} \mu_{C^{-1}(\Phi)}(x) \mu_{C_k}^{(1)}(C_j|\Phi) \\ &= \sum_{C_k \in \Phi} \mu_{C_i}^{(n)}(C_k|\Phi) (E_\Phi)_{k,j}. \end{aligned}$$

The Chapman-Kolmogorov equation follows directly from the theory of Markov processes. The stochastic matrix E_Φ is irreducible iff for every $i, j \in \mathbb{N}$ there is some $n \in \mathbb{N}$ such that $0 < ((E_\Phi)^n)_{i,j} = \mu_{C_i}^{(n)}(C_j|\Phi)$, which is equivalent to the existence of some $x \in \{0,1\}^n$ such that $C_i \xrightarrow{x} C_j$. \square

The next proposition collects some relations between the emulation Markov process and the corresponding set of computers. We assume that the reader is familiar with the basic vocabulary from the theory of Markov chains.

Proposition 3.11 (Irreducibility and Aperiodicity): Let $\Phi \subset \Xi$ be a set of computers.

- Φ is irreducible $\Leftrightarrow \Phi = \Phi^U \Leftrightarrow \Phi \subset \overline{\Phi^U}$.
- If Φ is connected and $\#\Phi^U \geq 2$, then $\overline{\Phi^U}$ is irreducible and branching.
- If Φ is branching, then we can define the *period* of $C \in \Phi$ as $d(C) := \text{GGT} \left\{ n \in \mathbb{N} \mid \mu_C^{(n)}(C|\Phi) > 0 \right\}$ (resp. ∞ if this set is empty). If $\Phi \subset \Xi$ is irreducible, then $d(C) = d(D) =: d < \infty$ for every $C, D \in \Phi$ holds true. In this case, d will be called the period of Φ , and if $d = 1$, then Φ is called *aperiodic*.

Proof. To prove the first equivalence, suppose that $\Phi \subset \Xi$ is irreducible, i.e. for every $C, D \in \Phi$ it holds $C \longrightarrow D$. Thus, Φ is connected and $C \in \Phi^U$, so $\Phi \subset \Phi^U$, and since always $\Phi^U \subset \Phi$, it follows that $\Phi = \Phi^U$. On the other hand, if $\Phi = \Phi^U$, then for every $C, D \in \Phi$ it holds $C \longrightarrow D$, since $C \in \Phi^U$. Thus, Φ is irreducible. For the second equivalence, suppose that Φ is irreducible, thus, $\Phi = \Phi^U \subset \overline{\Phi^U}$. If on the other hand $\Phi \subset \overline{\Phi^U}$, it follows in particular for every $C \in \Phi$ that $C \longrightarrow X$ for every $X \in \Phi$, so Φ is irreducible.

For the second statement, let $C, X \in \overline{\Phi^U}$ be arbitrary. By definition of $\overline{\Phi^U}$, it follows that there is some $V \in \Phi$ such that $V \longrightarrow X$, and it holds $C \longrightarrow V$, so $C \longrightarrow X$, and $\overline{\Phi^U}$ is irreducible. By Proposition 3.7 and $\#\overline{\Phi^U} \geq \#\Phi^U \geq 2$, $\overline{\Phi^U}$ must be branching. The third statement is well-known from the theory of Markov processes. \square

A basic general result about Markov processes now gives us the desired absolute computer probability - almost, at least:

Theorem 3.12 (Stationary Alg. Computer Probability): Let $\Phi \subset \Xi$ be branching, irreducible and aperiodic. Then, for every $C, D \in \Phi$, the limit ("computer probability")

$$\mu(D|\Phi) := \lim_{n \rightarrow \infty} \mu_C^{(n)}(D|\Phi)$$

exists and is independent of C . There are two possible cases:

- (1) The Markov process which corresponds to Φ is transient or null recurrent. Then,

$$\mu(D|\Phi) = 0 \text{ for every } D \in \Phi.$$

- (2) The Markov process which corresponds to Φ is positive recurrent. Then,

$$\mu(D|\Phi) > 0 \text{ for every } D \in \Phi, \text{ and } \sum_{D \in \Phi} \mu(D|\Phi) = 1.$$

In this case, the vector $\underline{\mu}_\Phi := (\mu(C_1|\Phi), \mu(C_2|\Phi), \dots)$ is the unique stationary probability eigenvector of E_Φ , i.e. the unique probability vector solution to $\underline{\mu}_\Phi \cdot E_\Phi = \underline{\mu}_\Phi$.

Note that we have derived this result under quite weak conditions — e.g. in contrast to classical algorithmic probability, we do not assume that our computers have prefix-free domain. Nevertheless, we are left with the problem to determine whether a given set Φ of computers is positive recurrent (case (2) given above) or not (case (1)).

The most interesting case is $\Phi = \Xi^U$, i.e. the set of computers that are universal in the sense that they can simulate every other computer without any restriction. This set is “large” — apart from universality, we do not assume any additional property like e.g. being prefix. By Proposition 3.11, Ξ^U is irreducible and branching. Moreover, fix any universal computer $U \in \Xi^U$ and consider the computer $V \in \Xi$, given by

$$V(x) := \begin{cases} \lambda & \text{if } x = \lambda, \\ V(s) & \text{if } x = 0 \otimes s, \\ U(s) & \text{if } x = 1 \otimes s. \end{cases}$$

As $V \xrightarrow{1} U$, we know that $V \in \Xi^U$, and since $V \xrightarrow{0} V$, it follows that $\mu_V^{(1)}(V) > 0$, and so $d(V) = 1 = d(\Xi^U)$. Hence Ξ^U is aperiodic.

So is Ξ^U positive recurrent or not? Unfortunately, the answer turns out to be negative: Ξ^U is transient. The idea to prove this is to construct a sequence of universal computers M_1, M_2, M_3, \dots such that each computer M_i emulates the next computer M_{i+1} with large probability, that is, the probability tends to one as i gets large. Thus, starting the random walk on, say, M_1 , it will with positive probability stay on this M_i -path forever and never return to any other computer. See also Figure 3 in the Introduction for illustration.

Theorem 3.13 (Markoff Chaney Virus):

Ξ^U is transient, i.e. there is no stationary algorithmic computer probability on the universal computers.

Proof. Let $U \in \Xi^U$ be an arbitrary universal computer with $U(\lambda) = 0$. We define another computer $M_1 \in \Xi$ as follows: If some string $s \in \{0, 1\}^*$ is supplied as input, then M_1

- splits the string s into parts $s_1, s_2, \dots, s_k, s_{tail}$, such that $s = s_1 \otimes s_2 \otimes \dots \otimes s_k \otimes s_{tail}$ and $|s_i| = i$ for every $1 \leq i \leq k$. We also demand that $|s_{tail}| < k + 1$ (for example, if $s = 101101101011$, then $s_1 = 1, s_2 = 01, s_3 = 101, s_4 = 1010$ and $s_{tail} = 11$),
- tests if there is any $i \in \{1, \dots, k\}$ such that $s_i = 0^i$ (i.e. s_i contains only zeros). If yes, then M_1 computes and outputs $U(s_{i+1} \otimes \dots \otimes s_k \otimes s_{tail})$ (if there are several i with $s_i = 0^i$, then it shall take the smallest one). If not, then M_1 outputs $1^k = \underbrace{1 \dots 1}_{k \text{ times}}$.

Let $M_2 := M_1 \xrightarrow{1}$, $M_3 := M_1 \xrightarrow{1 \otimes 11}$, $M_4 := M_1 \xrightarrow{1 \otimes 11 \otimes 111}$ and so on, in general $M_n := M_1 \xrightarrow{1^{1+2+\dots+(n-1)}}$ resp. $M_i \xrightarrow{1^i} M_{i+1}$. We also have $M_i \xrightarrow{0^i} U$, so $M_i \in \Xi^U$ for every $i \in \mathbb{N}$. Thus, the computers M_i are all universal. Also, since $M_i(\lambda) = M_1(1^{1+\dots+(i-1)}) = 1^{i-1}$, the computers M_i are mutually different from each other, i.e. $M_i \neq M_j$ for $i \neq j$. Now consider the computers $M_i \xrightarrow{s}$ for $|s| = i$, but $s \neq 0^i$. It holds $M_i(s \otimes x) = M_1(1 \otimes 11 \otimes \dots \otimes 1^{i-1} \otimes s \otimes x)$. The only property of s that affects the outcome of M_1 's computation is the property to be different from 0^i . But this property is shared

by the string 1^i , i.e. $M_1(1 \otimes 11 \otimes \dots \otimes 1^{i-1} \otimes s \otimes x) = M_1(1 \otimes 11 \otimes \dots \otimes 1^{i-1} \otimes 1^i \otimes x)$, resp. $M_i(s \otimes x) = M_i(1^i \otimes x)$ for every $x \in \{0, 1\}^*$. Thus, $(M_i \xrightarrow{s}) = (M_i \xrightarrow{1^i}) = M_{i+1}$ for every $0^i \neq s \in \{0, 1\}^i$, and so

$$\mu_{M_i}^{(i)}(M_{i+1} | \Xi^U) = 1 - 2^{-i} \text{ for every } i \in \mathbb{N}.$$

Iterated application of the Chapman-Kolmogorov equation (6) yields for every $n \in \mathbb{N}$

$$\begin{aligned} \mu_{M_1}^{(1+2+\dots+(n-1))}(M_n | \Xi^U) &\geq \mu_{M_1}^{(1)}(M_2 | \Xi^U) \cdot \mu_{M_2}^{(2)}(M_3 | \Xi^U) \\ &\quad \dots \cdot \mu_{M_{n-1}}^{(n-1)}(M_n | \Xi^U) \\ &= \prod_{i=1}^{n-1} (1 - 2^{-i}) \\ &> \prod_{i=1}^{\infty} (1 - 2^{-i}) = 0.2887 \dots \end{aligned}$$

With at least this probability, the Markov process corresponding to Φ will follow the sequence of computers $\{M_i\}_{i \in \mathbb{N}}$ forever, without ever returning to M_1 . (Note that also the intermediately emulated computers like $M_1 \xrightarrow{11}$ are different from M_1 , since $M_1(\lambda) = \lambda$, but $(M_1 \xrightarrow{11})(\lambda) \neq \lambda$.) Thus, the eventual return probability to M_1 is strictly less than 1. \square

In this proof, every computer M_{i+1} is a modified copy of its ancestor M_i . In some sense, M_1 can be seen as some kind of “computer virus” that undermines the existence of a stationary computer probability. The theorem’s name “Markoff Chaney Virus” was inspired by a fictitious character in Robert Anton Wilson’s “Illuminatus!” trilogy¹.

The set Ξ^U is in some sense too large to allow the existence of stationary algorithmic probability distribution. Yet, there exist computer sets Φ that are actually positive recurrent and thus have such a probability distribution; here is an explicit example:

Example 3.14 (A Positive Recurrent Computer Set): Fix an arbitrary string $u \in \{0, 1\}^*$ with $|u| \geq 2$, and let U be a universal computer, i.e. $U \in \Xi^U$, with the property that it emulates every other computer via some string that does not contain u as a substring, i.e.

$$\forall D \in \Xi \exists d \in \{0, 1\}^* : U \xrightarrow{d} D \text{ and } u \text{ not substring of } d.$$

If $C \in \Xi$ is any computer, define a corresponding computer $C_{u,U}$ by $C_{u,U}(x) = U(y)$ if $x = w \otimes u \otimes y$ and y does not contain u as a substring, and as $C_{u,U}(x) = C(x)$ otherwise (that is, if x does not contain u). The string u is a “synchronizing word” for the computer $C_{u,U}$, in the sense that any occurrence of u in the input forces $C_{u,U}$ to “reset” and to emulate U .

¹“The Midget, whose name was Markoff Chaney, was no relative of the famous Chaney of Hollywood, but people did keep making jokes about that. [...] Damn the science of mathematics itself, the line, the square, the average, the whole measurable world that pronounced him a bizarre random factor. Once and for all, beyond fantasy, in the depth of his soul he declared war on the “statutory ape”, on law and order, on predictability, on negative entropy. He would be a random factor in every equation; from this day forward, unto death, it would be civil war: the Midget versus the Digits...”

We get a set of computers

$$\Phi_{u,U} := \{C_{u,U} \mid C \in \Xi\}.$$

Whenever x does not contain u as a substring, it holds

$$C \xrightarrow{x} D \Rightarrow C_{u,U} \xrightarrow{x} D_{u,U}.$$

It follows that $V := U_{u,U}$ is a universal computer for $\Phi_{u,U}$. Thus $\Phi_{u,U}$ is connected, and it is easy to see that $\Phi_{u,U}^U = \Phi_{u,U}^U$ and $\#\Phi_{u,U}^U \geq 2$. According to Proposition 3.11, $\Phi_{u,U}^U$ is irreducible and branching. An argument similar to that before Theorem 3.13 (where it was proved that Ξ^U is aperiodic) proves that $\Phi_{u,U}^U$ is also aperiodic. Moreover, by construction it holds for every computer $C \in \Phi_{u,U}^U$ and $\ell := |u|$

$$\mu_C^{(\ell)}(V|\Phi_{u,U}^U) \geq 2^{-\ell}.$$

The Chapman-Kolmogorov equation (6) then yields

$$\begin{aligned} \mu_C^{(n+\ell)}(V|\Phi_{u,U}^U) &= \sum_{X \in \Phi_{u,U}^U} \mu_C^{(n)}(X|\Phi_{u,U}^U) \mu_X^{(\ell)}(V|\Phi_{u,U}^U) \\ &\geq 2^{-\ell} \sum_{X \in \Phi_{u,U}^U} \mu_C^{(n)}(X|\Phi_{u,U}^U) = 2^{-\ell}. \end{aligned}$$

Consequently, $\limsup_{n \rightarrow \infty} \mu_C^{(n)}(V|\Phi_{u,U}^U) \geq 2^{-\ell}$. According to Theorem 3.12, it follows that $\Phi_{u,U}^U$ is positive recurrent. In particular, $\mu(V|\Phi_{u,U}^U) \geq 2^{-|u|}$. Note also that $\#\Phi_{u,U}^U = \infty$, so we do not have the trivial situation of a finite computer set.

Obviously, the computer set $\Phi_{u,U}$ in the previous example depends on the choice of the string u and the computer U ; different choices yield different computer sets and different probabilities. In the next section, we will see in Theorem 4.8 that every positive recurrent computer set contains an unavoidable “amount of arbitrariness”, and this fact has an interesting physical interpretation.

Given any positive recurrent computer set Φ (as in the previous example), the actual numerical values of the corresponding stationary computer probability $\mu(\cdot|\Phi)$ will in general be noncomputable. For this reason, the following lemma may be interesting, giving a rough bound on stationary computer probability in terms of emulation complexity:

Lemma 3.15: Let $\Phi \subset \Xi$ be positive recurrent². Then, for every $C, D \in \Phi$, we have the inequality

$$2^{-K_C(D)} \leq \frac{\mu(D|\Phi)}{\mu(C|\Phi)} \leq 2^{K_D(C)}.$$

Proof. We start with the limit $m \rightarrow \infty$ in the Chapman-Kolmogorov equation (6) and obtain

$$\begin{aligned} \mu(D|\Phi) &= \sum_{U \in \Phi} \mu(U|\Phi) \mu_U^{(n)}(D|\Phi) \\ &\geq \mu(C|\Phi) \mu_C^{(n)}(D|\Phi) \end{aligned}$$

for every $n \in \mathbb{N}_0$. Next, we specialize $n := K_C(D)$, then $\mu_C^{(n)}(D|\Phi) \geq 2^{-n}$. This proves the left hand side of the inequality. The right hand side can be obtained simply by interchanging C and D . \square

²In the following, by stating that some computer set $\Phi \subset \Xi$ is positive recurrent, we shall always assume that Φ is also branching, irreducible and aperiodic.

IV. SYMMETRIES AND STRING PROBABILITY

The aim of this section is twofold: on the one hand, we will derive an alternative proof of the non-existence of a stationary computer probability distribution on Ξ^U (which we have already proved in Theorem 3.13). The benefit of this alternative proof will be to generalize our no-go result much further: it will supply us with an interesting physical interpretation why getting rid of machine-dependence must be impossible. We discuss this in more detail in Section V.

On the other hand, we would like to explore what happens for computer sets Φ that actually *are* positive recurrent. In particular, we show that such sets generate a natural algorithmic probability on the *strings* — after all, finding such a probability distribution was our aim from the beginning (cf. the Introduction). Actually, this string probability turns out to be useful in proving our no-go generalization. Moreover, it shows that the hard part is really to define computer probability — once this is achieved, string probability follows almost trivially.

Here is how we define string probability. While computer probability $\mu(C|\Phi)$ was defined as the probability of encountering C on a random walk on the Φ -tree, we analogously define the probability of a string s as the probability of getting the output s on this random walk:

Definition 4.1 (String Probability): Let $\Phi \subset \Xi$ be branching and let $C \in \Phi$. The n -step string probability of $s \in \{0,1\}^*$ is defined as the probability of arriving at output s on a random walk of n steps on the Φ -tree of C , i.e.

$$\mu_C^{(n)}(s|\Phi) := \sum_{x \in \{0,1\}^n \cap C^{-1}(\Phi): C(x)=s} \mu_{C^{-1}(\Phi)}(x).$$

Theorem 4.2 (Stationary Algorithmic String Probability): If $\Phi \subset \Xi$ is positive recurrent, then for every $C \in \Phi$ and $s \in \{0,1\}^*$ the limit

$$\begin{aligned} \mu(s|\Phi) &:= \lim_{n \rightarrow \infty} \mu_C^{(n)}(s|\Phi) \\ &= \sum_{U \in \Phi} \mu(U|\Phi) \mu_U^{(0)}(s|\Phi) = \sum_{U \in \Phi: U(\lambda)=s} \mu(U|\Phi) \end{aligned}$$

exists and is independent of C .

Proof. It is easy to see from the definition of n -step string probability that

$$\mu_C^{(n)}(s|\Phi) = \sum_{U \in \Phi: U(\lambda)=s} \mu_C^{(n)}(U|\Phi).$$

Taking the limit $n \rightarrow \infty$, Theorem 3.12 yields equality of left and right hand side, and thus existence of the limit and independence of C . \square

In general, $\mu(\cdot|\Phi)$ is a probability distribution on $\overline{\{0,1\}^*}$ rather than on $\{0,1\}^*$, i.e. the undefined string can have positive probability, $\mu(\infty|\Phi) > 0$, so $\sum_{s \in \{0,1\}^*} \mu(s|\Phi) < 1$.

We continue by showing a Chapman-Kolmogorov-like equation (analogous to Equation (6)) for the string probability. Note that this equation differs from the much deeper result of Theorem A.6 in the following sense: it describes a weighted average of probabilities $\mu_U^{(n)}(s|\Phi)$, and those probabilities do not only depend on the computer U (as in Theorem A.6), but also on the choice of the subset Φ .

Proposition 4.3 (Chapman-Kolmogorov for String Prob.):

If $\Phi \subset \Xi$ is positive recurrent, then

$$\mu_C^{(m+n)}(s|\Phi) = \sum_{U \in \Phi} \mu_C^{(m)}(U|\Phi) \mu_U^{(n)}(s|\Phi)$$

for every $C \in \Phi$, $m, n \in \mathbb{N}_0$ and $s \in \overline{\{0,1\}^*}$.

Proof. For $x, y \in \{0,1\}^*$, we use the notation

$$\delta_{x,y} := \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases}$$

and calculate

$$\begin{aligned} \mu_C^{(m+n)}(s|\Phi) &= \sum_{x \in \{0,1\}^{m+n} \cap C^{-1}(\Phi)} \mu_{C^{-1}(\Phi)}(x) \cdot \delta_{s,C(x)} \\ &= \sum_{U \in \Phi} \mu_C^{(m+n)}(U|\Phi) \mu_U^{(0)}(s|\Phi) \\ &= \sum_{U \in \Phi} \sum_{V \in \Phi} \mu_C^{(m)}(V|\Phi) \mu_V^{(n)}(U|\Phi) \mu_U^{(0)}(s|\Phi) \\ &= \sum_{V \in \Phi} \mu_C^{(m)}(V|\Phi) \sum_{U \in \Phi} \mu_V^{(n)}(U|\Phi) \mu_U^{(0)}(s|\Phi). \end{aligned}$$

The second sum equals $\mu_V^{(n)}(s|\Phi)$ and the claim follows. \square

For prefix computers C , algorithmic probability $P_C(s)$ of any string s as defined in Equation (1) and the expression $2^{-K_C(s)}$ differ only by a multiplicative constant [4]. Here is an analogous inequality for stationary string probability:

Lemma 4.4: Let $\Phi \subset \Xi$ be positive recurrent and $C \in \Phi$ some arbitrary computer, then

$$\mu(s|\Phi) \geq \mu(C|\Phi) \cdot 2^{-K_C(s)} \quad \text{for all } s \in \{0,1\}^*.$$

Proof. We start with the limit $m \rightarrow \infty$ in the Chapman-Kolmogorov equation given in Proposition 4.3 and get

$$\begin{aligned} \mu(s|\Phi) &= \sum_{U \in \Phi} \mu(U|\Phi) \mu_U^{(n)}(s|\Phi) \\ &\geq \mu(C|\Phi) \mu_C^{(n)}(s|\Phi) \end{aligned}$$

for every $n \in \mathbb{N}_0$. Then we specialize $n := K_C(s)$ and use $\mu_C^{(n)}(s|\Phi) \geq 2^{-n}$ for this choice of n . \square

Looking for further properties of stationary string probability, it seems reasonable to conjecture that, for many computer sets Φ , a string $s \in \{0,1\}^*$ (like $s = 10111$) and its inverse \bar{s} (in this case $\bar{s} = 01000$) have the same probability $\mu(s|\Phi) = \mu(\bar{s}|\Phi)$, since both seem to be in some sense algorithmically equivalent. A general approach to prove such conjectures is to study *output transformations*:

Definition 4.5 (Output Transformation σ):

Let $\sigma : \{0,1\}^* \rightarrow \{0,1\}^*$ be a computable permutation. For every $C \in \Xi$, the map $\sigma \circ C$ is itself a computer, defined by $\sigma \circ C(x) := \sigma(C(x))$. The map $C \mapsto \sigma \circ C$ will be called an *output transformation* and will also be denoted σ . Moreover, for computer sets $\Phi \subset \Xi$, we use the notation

$$\sigma \circ \Phi := \{\sigma \circ C \mid C \in \Phi\}.$$

Under reasonable conditions, string and computer probability are invariant with respect to output transformations:

Theorem 4.6 (Output Symmetry): Let $\Phi \subset \Xi$ be positive recurrent and closed³ with respect to some output transformation σ and its inverse σ^{-1} . Then, we have for every $C \in \Phi$

$$\mu(C|\Phi) = \mu(\sigma \circ C|\Phi)$$

and for every $s \in \{0,1\}^*$

$$\mu(s|\Phi) = \mu(\sigma(s)|\Phi).$$

Proof. Note that $\Phi = \sigma \circ \Phi$. Let $C, D \in \Phi$. Suppose that $C \xrightarrow{b} D$ for some bit $b \in \{0,1\}$. Then,

$$\sigma \circ C(b \otimes x) = \sigma(D(x)) = \sigma \circ D(x).$$

Thus, we have $\sigma \circ C \xrightarrow{b} \sigma \circ D$. It follows for the 1-step transition probabilities that

$$\begin{aligned} (E_\Phi)_{i,j} &= \mu_{C_i}^{(1)}(C_j|\Phi) \\ &= \mu_{\sigma \circ C_i}^{(1)}(\sigma \circ C_j|\Phi) = (E_{\sigma \circ \Phi})_{i,j} \end{aligned}$$

for every i, j . Thus, the emulation matrix E_Φ does not change if every computer C (or rather its number in the list of all computers) is exchanged with (the number of) its transformed computer $\sigma \circ C$ yielding the transformed emulation matrix $E_{\sigma \circ \Phi}$. But then, E_Φ and $E_{\sigma \circ \Phi}$ must have the same unique stationary probability eigenvector

$$\underline{\mu}_\Phi = (\mu(C_k|\Phi))_{k=1}^{\#\Phi} = \underline{\mu}_{\sigma \circ \Phi} = (\mu(\sigma \circ C_k|\Phi))_{k=1}^{\#\Phi}.$$

This proves the first identity, while the second identity follows from the calculation

$$\begin{aligned} \mu(s|\Phi) &= \sum_{U \in \Phi: U(\lambda)=s} \mu(U|\Phi) = \sum_{U \in \Phi: U(\lambda)=s} \mu(\sigma \circ U|\Phi) \\ &= \sum_{V \in \sigma \circ \Phi: V(\lambda)=\sigma(s)} \mu(V|\Phi) = \mu(\sigma(s)|\Phi). \quad \square \end{aligned}$$

Thus, if some computer set $\Phi \subset \Xi$ contains e.g. for every computer C also the computer \bar{C} which always outputs the bitwise inverse of C , then $\mu(s|\Phi) = \mu(\bar{s}|\Phi)$ holds. In some sense, this shows that the approach taken in this paper successfully eliminates properties of single computers (e.g. to prefer the string 10111 over 01000) and leaves only general algorithmic properties related to the set of computers.

Moreover, Theorem 4.6 allows for an alternative proof that Ξ^U and similar computer sets cannot be positive recurrent. We call a set of computable permutations $S := \{\sigma_i\}_{i \in \mathbb{N}}$ *cyclic* if every string $s \in \{0,1\}^*$ is mapped to infinitely many other strings by application of finite compositions of those permutations, i.e. if for every $s \in \{0,1\}^*$

$$\#\{\sigma_{i_1} \circ \sigma_{i_2} \circ \dots \circ \sigma_{i_N}(s) \mid N \in \mathbb{N}, i_n \in \mathbb{N}\} = \infty,$$

and if S contains with each permutation σ also its inverse σ^{-1} . Then, many computer subset cannot be positive recurrent:

Theorem 4.7 (Output Symmetry and Positive Recurrence):

Let $\Phi \subset \Xi$ be closed with respect to a cyclic set of output transformations, then Φ is not positive recurrent.

Proof. Suppose Φ is positive recurrent. Let $S := \{\sigma_i\}_{i \in \mathbb{N}}$ be the corresponding cyclic set of output transformations. Let

³We say that a computer set $\Phi \subset \Xi$ is *closed* with respect to some transformation $T : \Xi \rightarrow \Xi$ if $\Phi \supset T(\Phi) := \{T(C) \mid C \in \Phi\}$.

$s \in \{0, 1\}^*$ be an arbitrary string, then for every composition $\sigma := \sigma_{i_1} \circ \dots \circ \sigma_{i_N}$, we have by Theorem 4.6

$$\mu(s|\Phi) = \mu(\sigma(s)|\Phi).$$

Since S is cyclic, there are infinitely many such transformations σ , producing infinitely many strings $\sigma(s)$ which all have the same probability. It follows that $\mu(s|\Phi) = 0$. Since $s \in \{0, 1\}^*$ was arbitrary, this is a contradiction. \square

Again, we conclude that Ξ^U is not positive recurrent, since this computer set is closed with respect to *all* output transformations.

Although Ξ^U is not positive recurrent, there might be a unique, natural, “maximal” or “most interesting” subset $\Phi \subset \Xi$ which is positive recurrent. What can we say about this idea? In fact, the following theorem says that this is also impossible. As this theorem is only a simple generalization of Theorem 4.6, we omit the proof.

Theorem 4.8 (Non-Uniqueness): If $\Phi \subset \Xi$ is positive recurrent, then so is $\sigma \circ \Phi$ for every computable permutation (=output transformation) σ . Moreover,

$$\mu(C|\Phi) = \mu(\sigma \circ C|\sigma \circ \Phi)$$

for every $C \in \Phi$, and

$$\mu(s|\Phi) = \mu(\sigma(s)|\sigma \circ \Phi)$$

for every $s \in \overline{\{0, 1\}^*}$.

This means that there cannot be a unique “natural” positive recurrent computer set Φ : for every such set Φ , there exist output transformations σ such that $\sigma \circ \Phi \neq \Phi$ (this follows from Theorem 4.7). But then, Theorem 4.8 proves that $\sigma \circ \Phi$ is positive recurrent, too — and it is thus another candidate for the “most natural” computer set.

V. CONCLUSIONS AND INTERPRETATION

We have studied a natural approach to get rid of machine-dependence in the definition of algorithmic probability. The idea was to look at a Markov process of universal computers emulating each other, and to take the stationary distribution as a natural probability measure on the computers.

This approach was only partially successful: as the corresponding Markov process on the set of *all* computers is not positive recurrent and thus has no unique stationary distribution, one has to choose a subset Φ of the computers, which introduces yet another source of ambiguity.

However, we have shown (cf. Example 3.14) that there exist non-trivial, infinite sets Φ of computers that are actually positive recurrent and possess a stationary algorithmic probability distribution. This distribution has beautiful properties and eliminates at least some of the machine-dependence arising from choosing a single, arbitrary universal computer as a reference machine (e.g. Theorem 4.6). It gives probabilities for computers as well as for strings (Theorem 4.2), agrees with the average output frequency (Theorem A.6), and does not assume that the computers have any specific structural property like e.g. being prefix-free.

The second main result can be stated as follows: *There is no way to get completely rid of machine-dependence, neither in*

the approach of this paper nor in any other similar but different approach. To understand why this is true, recall that the main reason for our no-go result was the symmetry of computer probability with respect to *output transformations* $C \mapsto \sigma \circ C$, where σ is a computable permutation on the strings. This can be seen in two places:

- In Theorem 4.7, this symmetry yields the result that any computer set which is “too large” (like Ξ^U) cannot be positive recurrent.
- Theorem 4.8 states that if a set Φ is positive recurrent, then $\sigma \circ \Phi$ must be positive recurrent, too. Since in this case $\Phi \neq \sigma \circ \Phi$ for many σ , this means that there cannot be a unique “natural” choice of the computer set Φ .

Output transformations have a natural physical interpretation as “renaming the objects that the strings are describing”. To see this, suppose we want to define the complexity of the microstate of a box of gas in thermodynamics (this can sometimes be useful, see [4]). Furthermore, suppose we are only interested in a coarse-grained description such that there are only countably many possibilities what the positions, velocities etc. of the gas particles might look like. Then, we can encode every microstate into a binary string, and define the complexity of a microstate as the complexity of the corresponding string (assuming that we have fixed an arbitrary complexity measure K on the strings).

But there are always many different possibilities how to encode the microstate into a string (specifying the velocities in different data formats, specifying first the positions and then the velocities or the other way round etc.). If every encoding is supposed to be one-to-one and can be achieved by some machine, then two different encodings will always be related to each other by a computable permutation.

In more detail, if one encoding e_1 maps microstates m to encoded strings $e_1(m) \in \{0, 1\}^*$, then another encoding e_2 will map microstates m to $e_2(m) = \sigma(e_1(m))$, where σ is a computable permutation on the strings (that depends on e_1 and e_2). Choosing encoding e_1 , a microstate m will be assigned the complexity $K(e_1(m))$, while for encoding e_2 , it will be assigned the complexity $K(\sigma \circ e_1(m))$. That is, there is an unavoidable ambiguity which arises from the arbitrary choice of an encoding scheme. Switching between the two encodings amounts to “renaming” the microstates, and this is exactly an output transformation in the sense of this paper.

Even if we do not have the situation that the strings shall describe physical objects, we encounter a similar ambiguity already in the definition of a computer: a computer, i.e. a partial recursive function, is described by a Turing machine computing that function. Whenever we look at the output of a Turing machine, we have to “read” the output from the machine’s tape which can potentially be done in several inequivalent ways, comparable to the different “encodings” described above.

Every kind of attempt to get rid of those additive constants in Kolmogorov complexity will have to face this ambiguity of “renaming”. This is why we think that all those attempts must fail.

APPENDIX

STRING PROBABILITY IS THE WEIGHTED AVERAGE OF OUTPUT FREQUENCY

This appendix is rather technical and can be skipped on first reading. Its aim is to prove Theorem A.6. This theorem says that the string probability which has been introduced in Definition 4.1 in Section IV is exactly what we really wanted to have from the beginning: in the introduction, our main motivation to find a probability measure on the computers was to define machine-independent algorithmic probability of strings as the weighted mean over all universal computers as stated in Equation (3). Theorem A.6 says that string probability can be written exactly in this way, given some natural assumptions on the reference set of computers.

Note that Theorem A.6 is a surprising result for the following reason: string probability, as defined in Definition 4.1, only depends on the outputs of the computers on the “universal subtree”, that is, on the leaves in Figure 4 which correspond to bold lines. But output frequency, as given on the right-hand side in Theorem A.6 and defined in Definition 2.1, counts the outputs on *all* leaves — that is, output frequency is a property of a single computer, not of the computer subset that is underlying the emulation Markov process.

In Section IV, we have studied output transformations on computers — the key idea in this appendix will be to study *input transformations* instead. So what is an input transformation? If $\sigma : \{0, 1\}^* \rightarrow \{0, 1\}^*$ is a computable permutation on the strings and $C \in \Xi$ is some computer, we might consider the transformed computer $C \circ \sigma$, given by $(C \circ \sigma)(s) := C(\sigma(s))$. But this turns out not to be useful, since such transformations do not preserve the emulation structure. In fact, the most important and useful property of output transformations in Section IV was that they preserve the emulation structure: it holds

$$C \xrightarrow{s} D \iff \sigma \circ C \xrightarrow{s} \sigma \circ D.$$

But for transformations like $C \mapsto C \circ \sigma$, there is no such identity — hence we have to look for a different approach. It turns out that a successful approach is to look only at a restricted class of permutations, and also to introduce equivalence classes of computers:

Definition A.1 (Equivalence Classes of Computers):

For every $k \in \mathbb{N}$, two computers $C, D \in \Xi$ are called k -equivalent, denoted $C \stackrel{k}{\sim} D$, if $C(x) = D(x)$ for every $x \in \{0, 1\}^*$ with $|x| \geq k$. We denote the corresponding equivalence classes by $[C]_k$ and set

$$[\Phi]_k := \{[C]_k \mid C \in \Phi\}.$$

A computer set $\Phi \subset \Xi$ is called *complete* if for every $C \in \Phi$ and $k \in \mathbb{N}$ it holds $[C]_k \subset \Phi$. If $\Phi \subset \Xi$ is positive recurrent and complete, we set for every $[C]_k \in [\Phi]_k$

$$\mu([C]_k|\Phi) := \sum_{C \in [C]_k} \mu(C|\Phi).$$

It is easy to see that for every $C, D \in \Xi$ it holds

$$C \stackrel{k}{\sim} D \iff \left[\left(C \xrightarrow{s} \right) \stackrel{k}{\sim} \left(D \xrightarrow{s} \right) \text{ for every } s \in \{0, 1\}^* \right],$$

thus, the definition $\mu_{[C]_k}^{(n)}([D]_k|\Phi) := \sum_{D \in [D]_k} \mu_C^{(n)}(D|\Phi)$ makes sense for $n \in \mathbb{N}$ and $[C]_k, [D]_k \in [\Phi]_k$ and is independent of the choice of the representative $C \in [C]_k$. Enumerating the equivalence classes $[\Phi]_k = \{[C_1]_k, [C_2]_k, [C_3]_k, \dots\}$ in arbitrary order, we can define an associated emulation matrix $\mathcal{E}_{\Phi, k}$ as

$$(\mathcal{E}_{\Phi, k})_{i, j} := \mu_{[C_i]_k}^{(1)}([C_j]_k|\Phi).$$

It is easily checked that if Φ is positive recurrent, then the Markov process described by the transition matrix $\mathcal{E}_{\Phi, k}$ must also be irreducible, aperiodic and positive recurrent, and $\underline{\mu}_{\Phi, k} := (\mu([C_1]_k|\Phi), \mu([C_2]_k|\Phi), \mu([C_3]_k|\Phi), \dots)$ is the unique probability vector solution to the equation $\underline{\mu}_{\Phi, k} \cdot \mathcal{E}_{\Phi, k} = \underline{\mu}_{\Phi, k}$.

Now we can define input transformations:

Definition A.2 (Input Transformation \mathcal{I}_σ):

Let $\sigma : \{0, 1\}^n \rightarrow \{0, 1\}^n$ be a permutation such that there is at least one string $x \in \{0, 1\}^n$ for which $x_1 \neq \sigma(x)_1$, where x_1 denotes the first bit of x . For every $s \in \{0, 1\}^*$, let $\mathcal{I}_\sigma(s)$ be the string that is generated by applying σ to the last n bits of s (e.g. if $n = 1$, $\sigma(1) = 0$ and $s = 1011$, then $\mathcal{I}_\sigma(s) = 1010$). If $|s| < n$, then $\mathcal{I}_\sigma(s) := s$. For every $C \in \Xi$, the \mathcal{I}_σ -transformed computer $\mathcal{I}_\sigma(C)$ is defined by

$$(\mathcal{I}_\sigma(C))(s) := C(\mathcal{I}_\sigma(s)) \text{ for every } s \in \{0, 1\}^*.$$

We call $|\sigma| := n$ the *order* of σ . Moreover, we use the notation

$$\mathcal{I}_\sigma(\Phi) := \{\mathcal{I}_\sigma(C) \mid C \in \Phi\}.$$

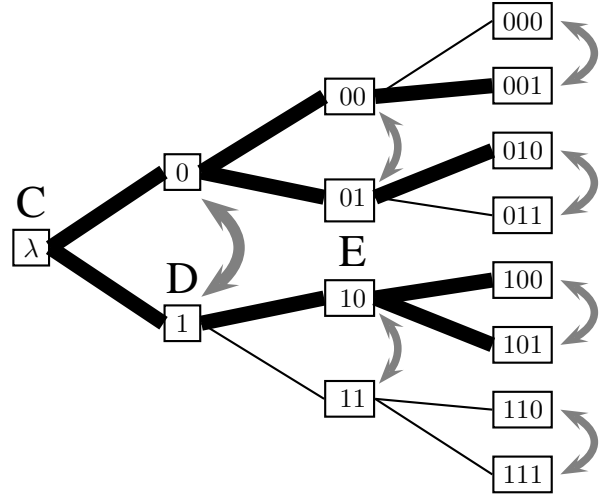


Fig. 5. The input transformation $C \mapsto \mathcal{I}_\sigma(C)$ for $\sigma(0) = 1, \sigma(1) = 0$.

The action of an input transformation is depicted in Figure 5: Changing e.g. the last bit of the input causes a permutation of the outputs corresponding to neighboring branches. As long as Φ is complete and closed with respect to that input transformation, the emulation structure will not be changed. This is a byproduct of the proof of the following theorem:

Theorem A.3 (Input Symmetry): Let $\Phi \subset \Xi$ be positive recurrent, complete and closed with respect to an input transformation \mathcal{I}_σ . Then, for every $k \geq |\sigma|$

$$\mu([C]_k|\Phi) = \mu([\mathcal{I}_\sigma(C)]_k|\Phi).$$

Proof. Suppose that $[C]_k \xrightarrow{0} [C_0]_k$, i.e. $C(0 \otimes x) = C_0(x)$ for every $|x| \geq k$, $C \in [C]_k$ and $C_0 \in [C_0]_k$. As $|\sigma| \leq k$,

$$\begin{aligned} (\mathcal{I}_\sigma(C))(0 \otimes x) &= C(\mathcal{I}_\sigma(0 \otimes x)) = C(0 \otimes \mathcal{I}_\sigma(x)) \\ &= C_0(\mathcal{I}_\sigma(x)) = \mathcal{I}_\sigma(C_0)(x), \end{aligned}$$

so $[\mathcal{I}_\sigma(C)]_k \xrightarrow{0} [\mathcal{I}_\sigma(C_0)]_k$. Analogously, from $[C]_k \xrightarrow{1} [C_1]_k$ it follows that $[\mathcal{I}_\sigma(C)]_k \xrightarrow{1} [\mathcal{I}_\sigma(C_1)]_k$ and vice versa. Thus,

$$(\mathcal{E}_{\Phi,k})_{i,j} = \mu_{[C_i]_k}^{(1)}([C_j]_k|\Phi) = \mu_{[\mathcal{I}_\sigma(C_i)]_k}^{(1)}([\mathcal{I}_\sigma(C_j)]_k|\Phi).$$

So interchanging every equivalence class of computers with its transformed class leaves the emulation matrix invariant. A similar argument as in Theorem 4.6 proves the claim. \square

We are now heading towards an analogue of Equation (3), i.e. towards a proof that our algorithmic string probability equals the weighted average of output frequency. This needs some preparation:

Definition A.4 (Input Symmetry Group): Let \mathcal{I}_σ be an input transformation of order $n \in \mathbb{N}$. A computer $C \in \Xi$ is called \mathcal{I}_σ -symmetric if $\mathcal{I}_\sigma(C) = C$ (which is equivalent to $[\mathcal{I}_\sigma(C)]_n = [C]_n$). The *input symmetry group* of C is defined as

$$\mathcal{I} - \text{SYM}(C) := \{\mathcal{I}_\sigma \text{ input transformation} \mid \mathcal{I}_\sigma(C) = C\}.$$

Every transformation of order $n \in \mathbb{N}$ can also be interpreted as a transformation on $\{0,1\}^N$ for $N > n$, by setting

$$\sigma(x_1 \otimes x_2 \otimes \dots \otimes x_N) := (x_1 \otimes \dots \otimes x_{N-n}) \otimes \sigma(x_{N-n+1}, \dots, x_N)$$

whenever $x_i \in \{0,1\}$. With this identification, $\mathcal{I} - \text{SYM}(C)$ is a group.

Proposition A.5 (Input Symmetry and Irreducibility): Let $\Phi \subset \Xi$ be irreducible. Then $\mathcal{I} - \text{SYM}(C)$ is the same for every $C \in \Phi$ and can be denoted $\mathcal{I} - \text{SYM}(\Phi)$.

Proof. Let $\Phi \subset \Xi$ be irreducible, and let $C \in \Phi$ be \mathcal{I}_σ -symmetric, i.e. $C(\mathcal{I}_\sigma(s)) = C(s)$ for every $s \in \{0,1\}^*$. Let $D \in \Phi$ be an arbitrary computer. Since Φ is irreducible, it holds $C \xrightarrow{} D$, i.e. there is a string $x \in \{0,1\}^*$ with $C(x \otimes s) = D(s)$ for every $s \in \{0,1\}^*$. Let $|\sigma| \geq |\sigma|$, then

$$D(s) = C(x \otimes s) = C(\mathcal{I}_\sigma(x \otimes s)) = C(x \otimes \mathcal{I}_\sigma(s)) = D(\mathcal{I}_\sigma(s))$$

and D is also \mathcal{I}_σ -symmetric. \square

For most irreducible computer sets like $\Phi = \Xi^U$, the input symmetry group will only consist of the identity, i.e. $\mathcal{I} - \text{SYM}(\Phi) = \{\text{Id}\}$.

Now we are ready to state the most interesting result of this section:

Theorem A.6 (Equivalence of Definitions): If $\Phi \subset \Xi$ is positive recurrent, complete and closed with respect to every input transformation \mathcal{I}_σ with $|\sigma| \leq n \in \mathbb{N}_0$, then

$$\mu(s|\Phi) = \sum_{U \in \Phi} \mu(U|\Phi) \mu_U^{(n)}(s) \text{ for every } s \in \overline{\{0,1\}^*},$$

where $\mu_U^{(n)}(s)$ is the output frequency as introduced in Definition 2.1.

Proof. The case $n = 0$ is trivial, so let $n \geq 1$. It is convenient to introduce another equivalence relation on the

computer classes. We define the corresponding equivalence classes (“transformation classes”) as

$$\{V\}_k := \{[X]_k \in [\Phi]_k \mid \exists \mathcal{I}_\sigma : |\sigma| \leq k, [\mathcal{I}_\sigma(V)]_k = [X]_k\}.$$

Thus, two computer classes $[X]_k$ and $[Y]_k$ are elements of the same transformation class if one is an input transformation (of order less than k) of the other. Again, we set $\{\Phi\}_k := \{\{X\}_k \mid X \in \Phi\}$.

For every $X \in [X]_n$, the probability $\mu_X^{(n)}(s|\Phi)$ is the same and can be denoted $\mu_{[X]_n}^{(n)}(s|\Phi)$. According to Proposition 4.3, we have

$$\mu(s|\Phi) = \sum_{\{X\}_n \in \{\Phi\}_n} \sum_{[Y]_n \in \{X\}_n} \mu([Y]_n|\Phi) \mu_{[Y]_n}^{(n)}(s|\Phi).$$

Due to Theorem A.3, the probability $\mu([Y]_n|\Phi)$ is the same for every $[Y]_n \in \{X\}_n$. Let $[X]_n$ be an arbitrary representative of $\{X\}_n$, then

$$\mu(\{X\}_n|\Phi) := \sum_{[Y]_n \in \{X\}_n} \mu([Y]_n|\Phi) = \#\{X\}_n \cdot \mu([X]_n|\Phi).$$

The two equations yield

$$\mu(s|\Phi) = \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{\#\{X\}_n} \sum_{[Y]_n \in \{X\}_n} \mu_{[Y]_n}^{(n)}(s|\Phi).$$

Let \mathbf{S}_{2^n} be the set of all permutations on $\{0,1\}^n$. Two permutations $\sigma_1, \sigma_2 \in \mathbf{S}_{2^n}$ are called Φ -equivalent if there exists a $\sigma \in \mathcal{I} - \text{SYM}(\Phi)$ such that $\sigma_1 = \sigma \circ \sigma_2$ (recall that Φ is irreducible). This is the case if and only if $\mathcal{I}_{\sigma_1}(C) = \mathcal{I}_{\sigma_2}(C)$ for one and thus every computer $C \in \Phi$. The set of all Φ -equivalence classes will be denoted $\mathbf{S}_n(\Phi)$. Every computer class $[Y]_n \in \{X\}_n$ is generated from $[X]_n$ by some input transformation. If X is an arbitrary representative of $[X]_n$, we thus have

$$\mu(s|\Phi) = \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{\#\{X\}_n} \sum_{[\sigma] \in \mathbf{S}_n(\Phi)} \mu_{[\mathcal{I}_\sigma(X)]_n}^{(n)}(s|\Phi),$$

where $\sigma \in [\sigma]$ is an arbitrary representative. For every equivalence class $[\sigma]$, it holds true $\#[\sigma] = \#(\mathbf{S}_{2^n} \cap \mathcal{I} - \text{SYM}(\Phi))$, thus

$$\begin{aligned} \mu(s|\Phi) &= \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{\#\{X\}_n} \cdot \frac{1}{\#(\mathcal{I} - \text{SYM}(\Phi) \cap \mathbf{S}_{2^n})} \\ &\quad \sum_{\sigma \in \mathbf{S}_{2^n}} \mu_{[\mathcal{I}_\sigma(X)]_n}^{(n)}(s|\Phi). \end{aligned}$$

By definition of the set $\mathbf{S}_n(\Phi)$,

$$\#\{X\}_n \cdot \#(\mathcal{I} - \text{SYM}(\Phi) \cap \mathbf{S}_{2^n}) = \#\mathbf{S}_{2^n} = (2^n)!.$$

Using that $\#\{X\}_n = \#\mathbf{S}_n(\Phi)$, we obtain

$$\begin{aligned} \mu(s|\Phi) &= \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{(2^n)!} \sum_{\sigma \in \mathbf{S}_{2^n}} \mu_{[\mathcal{I}_\sigma(X)]_n}^{(n)}(s|\Phi) \\ &= \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{(2^n)!} \sum_{\sigma \in \mathbf{S}_{2^n}} \sum_{x \in \{0,1\}^n} \delta_{\mathcal{I}_\sigma(X)(x),s} \mu_{X^{-1}(\Phi)}(x). \end{aligned}$$

As $|x| = n \geq |\sigma|$ it holds $\mathcal{I}_\sigma(X)(x) = X(\mathcal{I}_\sigma(x)) = X(\sigma(x))$. The substitution $y := \sigma(x)$ yields

$$\begin{aligned} \mu(s|\Phi) &= \sum_{\{X\}_n \in \{\Phi\}_n} \frac{\mu(\{X\}_n|\Phi)}{(2^n)!} \sum_{y \in \{0,1\}^n} \delta_{X(y),s} \\ &\quad \sum_{\sigma \in \mathbf{S}_{2^n}} \mu_{X^{-1}(\Phi)}(\sigma^{-1}(y)). \end{aligned}$$

Up to normalization, the rightmost sum is the average of all permutations of the probability vector $\mu_{X^{-1}(\Phi)}$, thus

$$\frac{1}{(2^n)!} \sum_{\sigma \in \mathbf{S}_{2^n}} \mu_{X^{-1}(\Phi)}(\sigma^{-1}(y)) = 2^{-n}.$$

Recall that X was an arbitrary representative of an arbitrary representative of $\{X\}_n$. The last two equations yield

$$\begin{aligned} \mu(s|\Phi) &= \sum_{\{X\}_n \in \{\Phi\}_n} \mu(\{X\}_n|\Phi) \sum_{y \in \{0,1\}^n} \delta_{X(y),s} 2^{-n} \\ &= \sum_{\{X\}_n \in \{\Phi\}_n} \sum_{[X]_n \in \{X\}_n} \mu([X]_n|\Phi) \mu_x^{(n)}(s) \\ &= \sum_{\{X\}_n \in \{\Phi\}_n} \sum_{[X]_n \in \{X\}_n} \sum_{X \in [X]_n} \mu(X|\Phi) \mu_x^{(n)}(s) \\ &= \sum_{X \in \Phi} \mu(X|\Phi) \mu_X^{(n)}(s). \end{aligned}$$

Note that if X and Y are representatives of representatives of an arbitrary transformation class $\{X\}_n$, then $\mu_X^{(n)}(s) = \mu_Y^{(n)}(s)$. \square

This theorem is the promised analogue of Equation (3): it shows that the string probability that we have defined in Definition 4.1 is the weighted average of output frequency as defined in Definition 2.1. For a discussion why this is interesting and surprising, see the first few paragraphs of this appendix.

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